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CONSTRUCTION OF THE UNDERGROUND CONTOUR OF THE FOUNDATION OF A HYDRAULIC STRUCTURE WITH CONSTANT-FLOW-VELOCITY PORTIONS IN THE PRESENCE OF A WATER-CONFINING LAYER

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A smooth underground contour of a hydraulic structure whose angles are rounded by the curves of constant filtration rate is constructed in the case where the water-permeable foundation is underlain by a water-confining layer consisting of horizontal and curvilinear portions which are characterized by a constant flow velocity, too. The corresponding boundary-value problem is solved by semiinverse application of the velocity-hodograph method first proposed by P. Ya. Polubarinova-Kochina and I. N. Kochina. The results of numerical calculations are given and a hydrodynamic analysis of the influence of the basic physical parameters of the model on the shape and dimensions of the underground contour of the dam and horizontal and curvilinear portions of the water-confining layer is made. The limiting cases where the water-confining layer is either horizontal or curvilinear throughout its length are studied in detail.

Keywords: *filtration, groundwater, hydraulic structure, velocity hodograph, conformal mappings.*

Introduction. The inverse approach to calculation of the underground contour of hydraulic structures was first used in [1, 2] where a smooth constant-velocity contour was constructed in the case where a water-permeable foundation was underlain by a horizontal water-confining layer. This work was started to further investigate of problems of such kind [3–7]. In [8], we did not only construct the smooth underground contour of a hydraulic structure but also determined the profile of the underlying curvilinear water-confining layer characterized by a constant flow velocity.

Below, we study a more general case where the water-confining layer consists of horizontal and curvilinear portions. The corresponding multiparametric mixed problem of analytical-function theory is solved using the semiinverse velocity-hodograph method [9–11]. The limiting cases of flow that are associated with the degeneration of the conformal-mapping parameters contained in the solution are noted: the P. Ya. Polubarinova-Kochina and I. N. Kochina case where the water-confining layer is either horizontal [1, 2] or curvilinear throughout its length [8].

Formulation of the Problem. We consider plane steady-state flow under the water-impermeable underground contour of a hydraulic structure $ABCC_1B_1A_1$ (Fig. 1). Let the contour of the foundation of the dam AA_1 consist of two vertical segments AB and A_1B_1 of equal length, the central horizontal segment CC_1, and the adjacent arcs of curves BC and B_1C_1 with a constant flow velocity $|w| = v_0$. The flow region is bounded from below by a water-confining layer G_1G consisting of a horizontal F_1F portion and curvilinear G_1F_1 and FG portions on which the filtration rate is constant $|w| = u_0$ $(0 \le u_0 < v_0)$, too. It is assumed that the boundaries of the headwater and tailwater are horizontal, the ground is homogeneous, and the motion obeys Darcy's law with a known filtration coefficient κ = const. The head acting on the structure H, the flow velocity v_0, the filtration flow rate Q, and the depth of the horizontal portion of the water-confining layer F_1F are assumed to be prescribed.

We introduce a complex potential of motion $\omega = \varphi + i\varphi$ (the range of variation in the variable ω is presented in Fig. 2) and a complex coordinate $z = x + iy$ referred respectively to kH and H. The problem is in determining the position of BC, B_1C_1, and G_1F_1 curves and F_1F and FG curves with the boundary conditions

$$A_1G_1 : y = 0\,,\quad \varphi = -0.5H\,;\quad A_1B_1 : x = -l\,,\quad \psi = Q\,;$$

$$C_1C : y = -d\,,\quad \psi = Q\,;\quad AB : x = l\,,\quad \psi = Q\,; \qquad (1)$$

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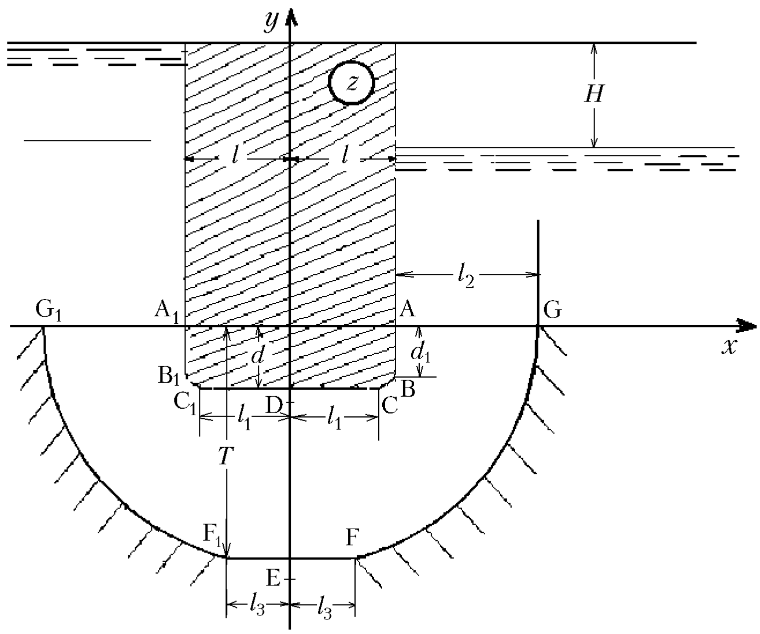

Fig. 1. Diagram of the underground contour of the hydraulic structure with constant-flow-velocity portions.

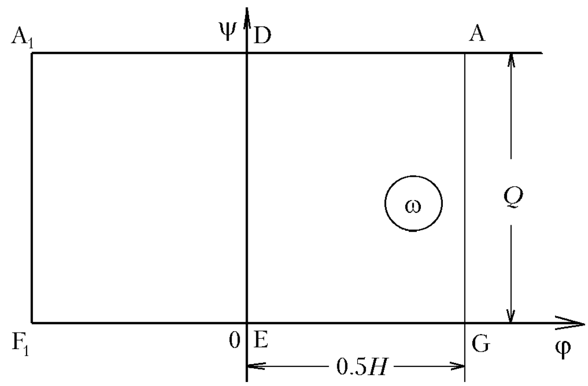

Fig. 2. Region of complex flow potential.

$$AG: y = 0,\quad \varphi = 0.5H;\quad FF_1: y = -T,\quad \psi = 0;\quad G_1F_1 \text{ and } FG: \psi = 0$$

so that the velocity of filtration along the curvilinear portions of the underground contour of the dam BC and B_1C_1 and the water-confining layer G_1G has constant values v_0 (prescribed) and u_0 (sought) respectively.

Construction of the Solution. It is convenient to take, as a canonical region [12], the rectangle $0 < \mathrm{Re}\,\tau < 0.5$ and $0 < \mathrm{Im}\,\tau < 0.5\rho$ of the plane τ (Fig. 3a), where $\rho(k) = K'/K$, $K' = K(k')$ and $k' = \sqrt{1-k^2}$; $K(k)$ is the complete elliptic integral of the first kind for the modulus k [13]. We now turn our attention to the region of a complex velocity w (Fig. 3b) corresponding to boundary conditions (1). This region differs from that for the case [8] only by the horizontal section F_1EF along the real semiaxis of the w plane; this makes it possible to use the Riemann–Schwartz principle of symmetry [14], which leads to a substantial reduction in unknown constants. Therefore, taking account of the total symmetry on the z, ω, and w planes, we restrict our consideration to the region of motion ABCDEFG (Fig. 1) and the corresponding like regions on the ω and w planes (Figs. 2 and 3b). Then, taking the coincidence of the region of the complex velocity $w = d\omega/dz$ (Fig. 3a) with that for the case [8] into account, we have

$$w(\tau) = v_0 \exp(\tau - 0.5)\pi i, \tag{2}$$

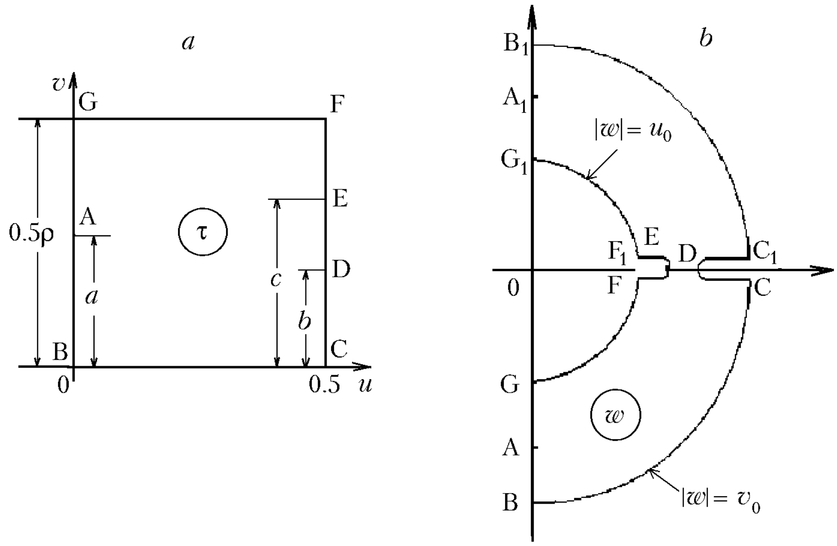

Fig. 3. Regions of auxiliary parametric variable (a) and complex velocity (b).

whence we determine the physical parameter $u_0 = v_0 \exp(-0.5\pi\rho)$.

We conformally map the rectangle of the auxiliary variable τ onto the region of the complex potential ω (Fig. 2). This yields

$$\omega = \frac{0.5}{K(k)} F\left[\arcsin \frac{\lambda}{n} \sqrt{\frac{1 - n^2 \operatorname{sn}^2(2K\tau, k)}{1 - \lambda^2 \operatorname{sn}^2(2K\tau, k)}}, m\right]. \tag{3}$$

Here $F(\varphi, m)$ is the elliptic integral of the first kind [13] for the modulus m, where

$$m = k\sqrt{\frac{(1 - k'^2A^2B^2)(1 - k'^2C^2)}{(1 - k'^2B^2)(1 - k'^2A^2C^2)}}; \quad \lambda = \sqrt{1 - k'^2B^2}; \quad n = \sqrt{1 - k'^2C^2};$$

$$A = \operatorname{sn}(2Ka, k'); \quad B = \operatorname{sn}(2Kb, k'); \quad C = \operatorname{sn}(2Kc, k');$$

$\operatorname{sn}(\varphi, k)$, $\operatorname{cn}(\varphi, k)$, and $\operatorname{dn}(\varphi, k)$ are the elliptic Jacobian functions (sine, cosine, and delta respectively) for the modulus k. The filtration flow rate is determined from the formula

$$Q = 0.5H\rho(m) = \frac{0.5HK'(m)}{K(m)}. \tag{4}$$

To solve the problem we use the first variant of the velocity-hodograph technique ([9, pp. 250–251], [10, p. 60], and [11, pp. 603–606]). Taking relations (2) and (3) into account and acting analogously to [15, 16], we arrive at the dependences

$$\frac{d\omega}{d\tau} = -\frac{M \operatorname{sn}(2K\tau, k) \operatorname{cn}(2K\tau, k) \operatorname{dn}(2K\tau, k)}{\Delta(\tau)},$$

$$\frac{dz}{d\tau} = -\frac{M \operatorname{sn}(2K\tau, k) \operatorname{cn}(2K\tau, k) \operatorname{dn}(2K\tau, k) \exp((0.5 - \tau)\pi i)}{v_0 \Delta(\tau)}, \tag{5}$$

$$\Delta(\tau) = \sqrt{[1 - \lambda^2 \operatorname{sn}^2(2K\tau, k)][1 - n^2 \operatorname{sn}^2(2K\tau, k)][A^2 + (1 - A^2)\operatorname{sn}^2(2K\tau, k)]}.$$

It can be checked that the functions (5) satisfy boundary conditions (1) formulated in terms of the functions $d\omega/d\tau$ and $dz/d\tau$ and thereby are the parametric solution of the initial boundary-value problem. Writing representations (5) for different portions of the boundary of the region τ followed by integration over the entire contour of the auxiliary region leads to a closing of the contour of the region of motion z and thereby serves as a computation control.

This yields the expressions for the basic geometric and filtration characteristics

$$\frac{M}{v_0}\int_0^{0.5} X_{BC}dt = \Delta l\,,\quad \frac{M}{v_0}\int_0^{0.5} Y_{BC}dt = \Delta d\,,$$

$$m\left(\int_0^{0.5\rho}\Phi_{EF}dt + \int_0^{0.5}\Phi_{FG}dt\right) = 0.5H\,,\quad \frac{M}{u_0}\int_0^{0.5} Y_{FG}dt = T\,, \tag{6}$$

the coordinates of points of the underground contour of the apron BC

$$x_{BC}(t) = l - \frac{M}{v_0}\int_0^t X_{BC}dt\,,\quad y_{BC}(t) = -d_1 - \frac{M}{v_0}\int_0^t Y_{BC}dt\,,\quad 0 \le t \le 0.5\,, \tag{7}$$

and the coordinates of the curvilinear water-confining layer FG

$$x_{FG}(t) = L - \frac{M}{u_0}\int_0^t X_{FG}dt\,,\quad y_{FG}(t) = -\frac{M}{u_0}\int_0^t Y_{FG}dt\,,\quad 0 \le t \le 0.5\,. \tag{8}$$

Here we have

$$\Delta l = l - l_1\,,\quad \Delta d = d - d_1\,,\quad L = l + l_2\,,$$

$$X_{BC} = \sin\pi t\,\frac{\mathrm{sn}\,(2Kt, k)\,\mathrm{cn}\,(2Kt, k)\,\mathrm{dn}\,(2Kt, k)}{\Delta(t)}\,,\quad Y_{BC} = \cos\pi t\,\frac{\mathrm{sn}\,(2Kt, k)\,\mathrm{cn}\,(2Kt, k)\,\mathrm{dn}\,(2Kt, k)}{\Delta(t)}\,,$$

$$\Phi_{EF} = \frac{\mathrm{sn}\,(2Kt, k')\,\mathrm{cn}\,(2Kt, k')}{\Delta_1(t)}\,,\quad \Phi_{FG} = k\,\frac{\mathrm{cn}\,(2Kt, k)\,\mathrm{dn}\,(2Kt, k)}{\Delta_2(t)}\,,\quad \Phi_{AG} = \frac{\mathrm{sn}\,(2Kt, k)\,\mathrm{dn}\,(2Kt, k)}{\Delta_3(t)}\,,$$

$$X_{FG} = \sin\pi t\,\Phi_{FG}\,,\quad Y_{FG} = \cos\pi t\,\Phi_{FG}\,,$$

$$\Delta_1(t) = \sqrt{[\mathrm{sn}^2(2Kt, k) - B^2]\,[\mathrm{sn}^2(2Kt, k) - C^2]\,[1 - k'^2A^2\mathrm{sn}^2(2Kt, k)]}\,,$$

$$\Delta_2(t) = \sqrt{[\mathrm{dn}^2(2Kt, k) - \lambda^2]\,[n^2 - k^2\,\mathrm{sn}^2(2Kt, k)]\,[1 - A^2\mathrm{dn}^2(2Kt, k)]}\,,$$

$$\Delta_3(t) = \sqrt{[1 - \lambda'^2\,\mathrm{sn}^2(2Kt, k)]\,[1 - n'^2\,\mathrm{sn}^2(2Kt, k)]\,[\mathrm{sn}^2(2Kt, k) - A^2]}\,.$$

Setting $t = 0.5$ in Eqs. (7) and (8), we find the sought dimensions of the underground contour of the apron and the water-confining layer

$$l_1 = x_{BC}(0.5)\,,\quad d_1 = y_{BC}(0.5)\,,\quad l_3 = L - x_{FG}(0.5)\,,\quad l_2 = \frac{M}{v_0}\int_a^{0.5\rho}\Phi_{AG}\exp(\pi t)\,dt. \tag{9}$$

Computations can be controlled by other expressions for the flow rate Q and the geometric dimensions l_2 and T:

$$l_1 = \frac{M}{v_0}\int_0^b X_{CD} dt\,, \quad d_1 = \frac{M}{v_0}\int_0^a Y_{AB} dt\,, \quad Q = M\int_0^{0.5\rho} \psi_{AG} dt\,, \quad Q = M\int_b^c \psi_{DE} dt\,,$$

$$l_3 = \frac{M}{v_0}\int_0^{0.5\rho} X_{EF} dt\,, \quad T = d + \frac{M}{v_0}\int_b^{0.5\rho} Y_{DE} \exp(\pi t)\, dt\,, \tag{10}$$

where

$$X_{CD} = \frac{\text{sn}(2Kt, k)\,\text{cn}(2Kt, k)\exp(\pi t)}{\Delta_4(t)}\,; \quad Y_{AB} = \frac{\text{sn}(2Kt, k)\,\text{dn}(2Kt, k)\exp(\pi t)}{\Delta_5(t)}\,;$$

$$\psi_{AG} = \frac{\text{sn}(2Kt, k)\,\text{dn}(2Kt, k)}{\Delta_3(t)}\,; \quad \psi_{DE} = \frac{\text{sn}(2Kt, k)\,\text{cn}(2Kt, k)}{\Delta_6(t)}\,; \quad X_{EF} = \frac{\text{sn}(2Kt, k)\,\text{cn}(2Kt, k)\exp(\pi t)}{\Delta_7(t)}\,;$$

$$\Delta_4(t) = \sqrt{[B^2 - \text{sn}^2(2Kt, k')]\,[C^2 - \text{sn}^2(2Kt, k')]\,[1 - k'^2 A^2\,\text{sn}^2(2Kt, k')]}\,;$$

$$\Delta_5(t) = \sqrt{[1 - \lambda'^2\,\text{sn}^2(2Kt, k')]\,[1 - n'^2\,\text{sn}^2(2Kt, k')]\,[A^2 - \text{sn}^2(2Kt, k')]}\,;$$

$$\Delta_6(t) = \sqrt{[\text{sn}^2(2Kt, k') - B^2]\,[C^2 - \text{sn}^2(2Kt, k')]\,[1 - k'^2 A^2\,\text{sn}^2(2Kt, k')]}\,;$$

$$\Delta_7(t) = \sqrt{[\text{sn}^2(2Kt, k') - B^2]\,[\text{sn}^2(2Kt, k') - C^2]\,[1 - k'^2 A^2\,\text{sn}^2(2Kt, k')]}\,.$$

Limiting Cases. 1. First of all we dwell on the case [1, 2] where the water-confining layer is horizontal throughout its length. The points G and F merge at infinity in the plane of motion z, and the rectangle of the plane of the auxiliary variable τ becomes a half-band $0 < \text{Re}\,\tau < 0.5$ and $0 < \text{Im}\,\tau < \infty$ (Fig. 3a), since the modulus k is equal to 0, $k' = 0$, $K = 0.5\pi$, $K' = \infty$, and consequently $\rho = \infty$. The solution for this limiting case is obtained from formulas (5)–(10) if we set $k = 0$ in them and take into account that for such a modulus, the elliptic functions degenerate into trigonometric ones: sn $(2K\tau, 0) = \sin \pi\tau$, cn $(2K\tau, 0) = \cos \pi\tau$, and dn $(2K\tau, 0) = 1$. From formulas (9) and (10), it follows that we have $l_2 = \infty$ and $l_3 = \infty$ and expressions (6) for H and T can be integrated in explicit form

$$H = \frac{2MK(k)}{\pi\sqrt{(1 - C^2)(1 - A^2B^2)}}\,, \quad T = \frac{M}{v_0\sqrt{(1 - A^2)(1 - B^2)(1 - C^2)}}\,, \tag{11}$$

where

$$k = \sqrt{\frac{(1 - A^2B^2)(1 - C^2)}{(1 - A^2C^2)(1 - B^2)}}\,.$$

Formulas (11) agree with the corresponding expressions in [2] (p. 191, formulas (7.17) and (7.18)), if we take into account that the parameters α and β from [2] are related to the parameters A, B, and C by the following relations:

$$\alpha = \frac{B}{kC}\sqrt{\frac{(1 - C^2)}{(1 - B^2)}}\,, \quad \beta = \frac{1}{k}\sqrt{\frac{(1 - C^2)}{(1 - B^2)}}\,.$$

2. The other limiting case is obtained from the general scheme if points E and F in the flow region merge i.e., when the horizontal impermeable portion is absent and the water-confining layer turns out to be curvilinear

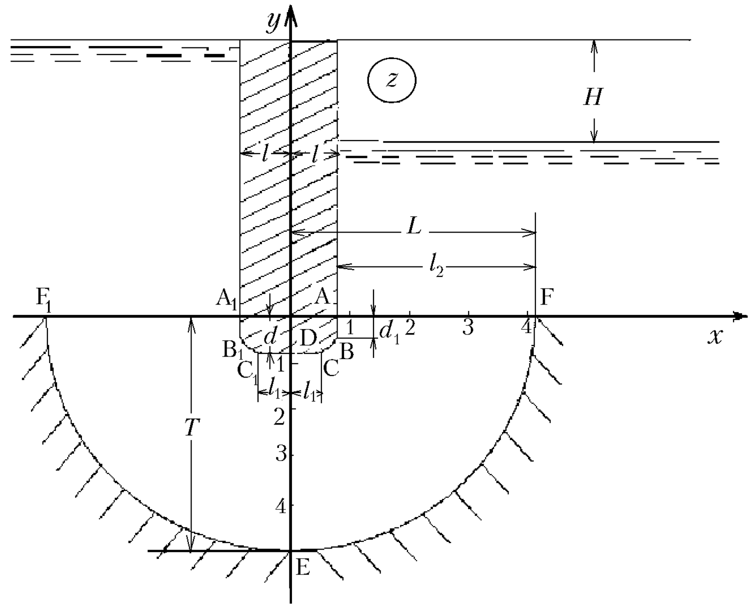

Fig. 4. Underground contour of the hydraulic structure, calculated for $H = 2$, $v_2 = 1$, $Q = 1.14$, $\Delta l = 0.296$, and $\Delta d = 0.295$.

throughout its length [8]. In this case we have the parameter $c = 0.5\rho$ in the plane τ and the solution is obtained from formulas (5)–(10), if we set $C = 1$ in them. As follows from (9)–(10), we have $l_3 = 0$.

Computational Scheme and Analysis of Numerical Results. Representations (5)–(10) contain five unknown constants A, B, C, k, and M. Relation (4) serves for determination of the modulus of elliptic integrals k. By virtue of the asymptotics [13]

$$\frac{K'}{K} = (\pi/2) \ln \left(4k'^{-1}\right)$$

the right-hand side of Eq. (4) cannot be prescribed in an arbitrary manner and is limited to a certain range of its variation which is determined by the critical values Q_* and H_* corresponding to the cases $k \approx 0$ and $k \approx 1$. Thus, the relation (4) regulates prescription of the physical parameters Q and H and consequently the area of applicability of the adopted flow diagram. It is noteworthy that an analogous situation occurred in a similar model [17]. The other three parameters of mapping A, B, and C ($0 < A < 1$ and $0 < B < C \leq 1$) are determined from Eqs. (6) for prescribed Δl, Δd, and T; the modeling constant M is pre-found from the third equation of (6) that fixes the acting head H. On determination of the unknown constants, we subsequently find the sought parameters of the underground contour of the structure l_1 and d_1, l_2 and l_3 from formulas (9) and the width and depth of the dam $l = l_1 + \Delta l$ and $d = d_1 + \Delta d$; finally, from formulas (7) and (8), we calculate respectively the coordinates of points of the underground contour of the structure BC and the curvilinear part FG.

We evaluate the influence of the model's physical parameters H, v_0, Q, Δl, and Δd on the shape and dimensions of the underground contour of the hydraulic structure from consideration of the case where there is no horizontal portion of the water-confining layer, i.e., for $l_3 = 0$ (Fig. 4).

Figure 5 plots d_1, l_2 (solid curves) and l_1, T (dot-dash curves) versus the parameters v_0, Q, Δl, and Δd.

An analysis of these plots enables us to draw the following conclusions. Decrease in the flow velocity and increase in the head acting on the structure lead to growth in the dimensions of the dam l, l_1, d, and d_1 and, conversely, to decrease in the dimensions of the curvilinear water-confining layer l_2 and T. The quantities l_1 and d_1 and consequently the width and thickness of the dam can be very great: the plots of Fig. 5 I show that a change of 1.5 times in the velocity increases the width l_1 and the thickness d_1 by 329 and 380.4% respectively. It is noticeable that the width l and the thickness d of the dam and the dimensions of the water-confining layer l_2 and T vary within 24–72%. From the plots given in Fig. 5 III and IV, it is seen that the dependences of d_1 and l_1 on the flow velocity and

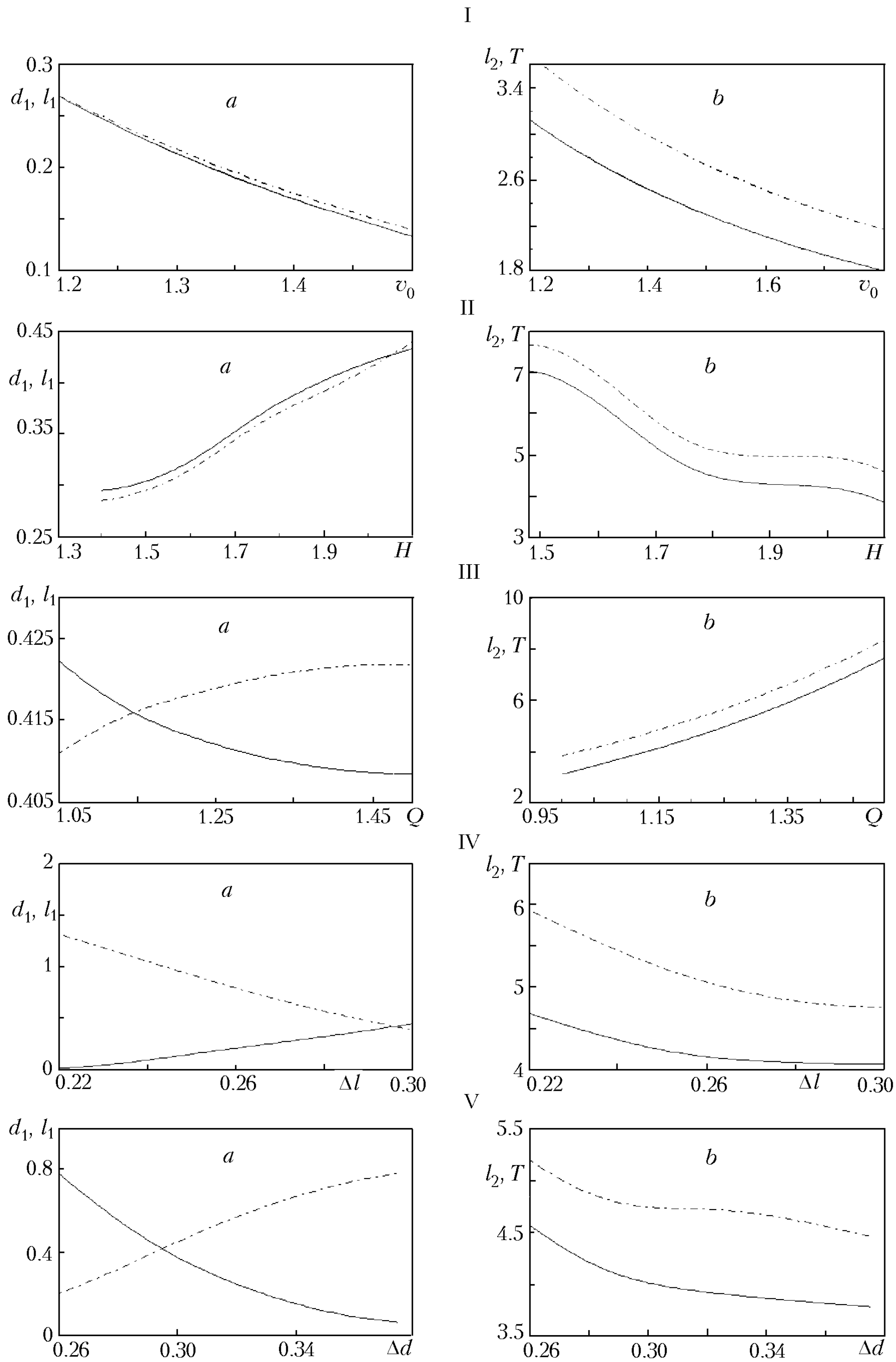

Fig. 5. Quantities d_1, l_1 (a) and T, l_2 (b) vs. v_0 (I) at constant H = 2, Q = 1.14, Δl = 0.296, and Δd = 0.295, vs. H (II) at constant v_0 = 1, Q = 1.14, Δl = 0.296, and Δd = 0.295, vs. Q (III) at constant v_0 = 1, H = 2, Δl = 0.296, and Δd = 0.295, vs. Δl (IV) at constant v_0 = 1, H = 2, Q = 1.14, and Δd = 0.295, and vs. Δd (V) at constant v_0 = 1, H = 2, Q = 1.14, and Δl = 0.296.

the acting head and of T and l_2 on all determining parameters of the model are qualitatively similar; for fixed values of v_0, Q, Δl, and Δd, the depth of the water-confining layer T exceeds the width l_2 by 10–20%, on the average.

Variation of the parameter Q produces only slight changes in the dimensions l, l_1, d, and d_1 (within 1.02–1.05 times), so that the influence of the filtration flow rate has virtually no effect on the dimensions of the dam.

Noteworthy is the same qualitative character of the dependences of l, l_1, d, and d_1 on the parameters Q and Δl: increase in the filtration flow rate and in the difference Δl leads to growth in the dam thickness d (and in the quantity d_1) and to decrease in the dam width l (and in the quantity l_1). At the same time, the character of change in the dimensions of the dam with variation of the quantities Δl and Δd is just the reverse. The plots of Fig. 5 IV and V that refer to these parameters reflect the regularity which is natural from the physical viewpoint: increase in the difference Δl (Δd) is accompanied by decrease (growth) in the dam width l and by growth (decrease) in its thickness d. Thus, as Δl changes by 50%, the width l_1 decreases 4.1 times and the thickness d_1 increases 110.5 time; such a change in the quantity Δd leads to an increase of 5.2 time in the width l_1 and a decrease of 14.7 times in the thickness d_1.

As the parameters v_0, Q, Δl, and Δd grow, both the depth of the water-confining layer T and the width l_2 decrease (even if only slightly, within 1.1–1.7 times) and increase with filtration flow rate, and substantially: by 147 and 119% respectively. The quantities T and l_2 can be very great and can exceed not only the parameters d_1 and l_1 but also the dimensions themselves of the dam d and l respectively in the cases in question. Thus, it follows from the plot of Fig. 5 I that for $H = 1.4$, we have $l_1 = 0.285$, $l = 0.581$, and $l_2 = 6.678$, and consequently $l_2/l = 11.5$, whereas the plot of Fig. 5 V yields that for $\Delta l = 0.2$, we obtain $d_1 = 0.004$, $d = 0.299$, and $T = 6.495$; consequently, $T/d = 21.7$. Thus, the dimensions l_2 and T exceed the width of the dam l and its thickness d by 1049 and 2072% respectively.

Conclusions. We have obtained the exact analytical solution of the problem on construction of a smooth underground contour of a hydraulic structure whose angles are rounded by constant-velocity curves in the case where a water-permeable foundation is underlain by a water-confining layer consisting of horizontal and curvilinear portions. It has been established by numerical calculations that increase in the head acting on the structure and decrease in the flow velocity lead to growth in all dimensions of the dam and, conversely, to decrease in the dimensions of the curvilinear water-confining layer.

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NOTATION

a, b, and c, unknown affixes (images) of points A, D, and E on the plane of the auxiliary parametric variable; d, depth of the hydraulic structure; d_1, depth of the rectangular part of the structure; F, elliptic integral of the first kind for the modulus k; i, imaginary unit; Im, imaginary part of the complex number; H, acting head; k, modulus of elliptic integrals; k', additional modulus; K and K', complete elliptic integrals of the first kind for the modulus k and k' respectively; l, halfwidth of the apron; l_1, halfwidth of the horizontal underwater part of the dam; l_2, width of the water-permeable portion of outflow of water into the tailwater; l_3, width of the horizontal portion of the water-confining layer; L, distance from the origin of coordinates to the point G of the water-confining layer; m, modulus of elliptic integrals; M, scaling constant of modeling; n and n', auxiliary quantities related to the moduli k and k' and the parameter of mapping C; Q, filtration flow rate of water; Re, real part of the complex number; t, variable of integration; T, depth of the horizontal part of the water-confining layer; u_0 and v_0, velocities of filtration along the water-confining layer and the hydraulic structure respectively; w, complex flow velocity; x and y, abscissa and ordinate of a point of the flow region respectively; z, complex coordinate of a point of the flow region; κ, filtration coefficient; λ and λ', auxiliary quantities related to the moduli k and k' and the parameter of mapping B; ρ, dimensionless quantity related to the ratio of the complete elliptic integrals of the first kind K' and K; τ, auxiliary variable; φ, velocity potential; ψ, steam function; ω, complex flow potential.

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